On the chaotic aspects of highly excited string amplitudes

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Motivations

- H-P model of String-BH correspondence (arXiv :hep-th/9612146)
- The inclusion of the full string degeneracy in S-matrix elements, B-F (arXiv :1902.07016)
- Erratic behaviors of HES, G-R (arXiv :2103.15301)
- Black body radiation in string decays A-R (arXiv :hep-th/9901092)
- Probing of the string size and its shape in scattering processes, M-S (Nucl. Phys. B **349**)

Plan

Chaos in HES processes

- The general picture of chaos in HES
- Analysis of 3/4-point processes

Thermal effects from HES processes

- The general picture of HES as a thermodynamical system
- Thermalization in HES decay

BH horizon from HES

- HES form factor, size and shape of the HES as a BH horizon

Highly excited strings (HES)

Let's consider D dimensional bosonic string in the Fock space representation

$$\mathcal{H}_{str} \Rightarrow \widehat{M}^2_N = 2(\widehat{N} - a)\,, \quad \widehat{N} = \sum_n \alpha_{-n}{\cdot}\alpha_n\,, \quad a = \frac{D-2}{24}$$

spectrum

- $M^2 = 2(N-1)$, for each level N there are $\sum_n n g_n = N$ states $|\{n\},\{g_n\}\rangle$.
- The number of states at fixed N corresponds to the integer partitions of N
- The degeneracy of states $d_N \simeq e^{\sqrt{N}}$ and the entropy $\simeq \sqrt{N}$

state-operator correspondence

From the Heisenberg algebra of $j = D-2$ oscillators α^j_n : $[\alpha^j_n, \alpha^k_m] = n\,\delta^{jk}\,\delta_{m+n,0}$

$$\prod_{n=1}(\alpha^{j_n}_{-n})^{g_n}|0,p\rangle = |\{n\},\{g_n\},p\rangle^{\{j_n\}} \leftrightarrow VO(R^{(j_n)}_{dof}, p_N)$$

HES : BRST vertex operators

The general structure of VO

$$VO(R, p_N) = R_{dof} \int dz\, \mathcal{O}[\partial X]\, e^{ip_N \cdot X}(z)$$

S-matrix elements are computed with CFT technology applied to vertex operators

$$S(out \,|\, in) \simeq \Big\langle \prod_{\ell}^{in} VO(R_{dof}^{(j_\ell)}, p_{N_\ell}) \prod_{m}^{out} VO(R_{dof}^{(j_m)}, p_{N_m}) \Big\rangle$$

Physical VO are constrained by BRST symmetry

$$[Q_{BRST}, VO(R_{dof}^{(j_\ell)}, p_N)] = 0 \Rightarrow E_\ell(R_{dof}^{(j_\ell)}, p_N) = 0$$

there is a complicated set of ℓ-equations at each level N, that constrains :

- conformal structure :
  $\Delta_{VO} = 0 \Rightarrow M^2 = 2(\Delta_{\mathcal{O}} - 1)$ Mass shell condition of $\mathcal{H}_{string}$
- Lorentz structure of R_{dof} :
  - Irrep of massless little group $SO(D-2)$ if $N = 1$
  - Irrep of massive little group $SO(D-1)$ if $N > 1$

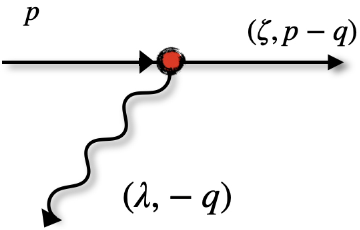
HES : Di Vecchia-Del Giudice-Fubini (DDF) setup

- In the DDF setup only physical d.o.f. are considered and VO are automatically BRST invariant !
- the isomorphism is provided by $\alpha_n^{j_n} \sim A_n^j = \oint_C \frac{dz}{2\pi} \partial X^i \, e^{inq\cdot X}(z)$
- state-operator correspondence is realized directly in terms of explicit conformal and Lorentz structure

DDF realization of BRST invariant vertex operators

Starting from a tachyonic vacuum $|0,p\rangle$, $p^2 = 2$ and using the constraints $q^2 = A_n \cdot q = 0$, $p \cdot q = 1$ one can obtain all the physical VOs

- Level $N = 1$, $\lambda \cdot A_{-1}|0,p\rangle|_{OPE} = \zeta(p,q)\cdot\partial X \, e^{i(p-q)X}$ with $(p-q)^2 = k^2 = 0$



<u>Lorentz covariant map of the polarization</u>

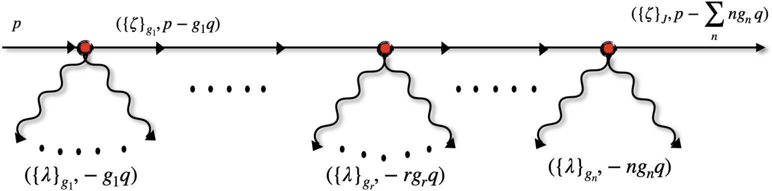
$$\lambda^j \Rightarrow \zeta^M(p,q) = \lambda_j \delta^{jM} - \lambda \cdot p \, q^M$$

With constraints

$$\zeta \cdot p = \zeta \cdot q = 0$$

Iterating the OPE one obtains the VO of an arbitrary highly excited state

$$\prod_n (\lambda_n\cdot A_{-n})^{g_n}|0,p\rangle = \sum_{a=0}^{[J/2]}\sum_{\pi\in\widehat{S}_J}\prod_{\ell=1}^{a}\zeta_{i_{\pi(2\ell+1)}}\cdot\zeta_{i_{\pi(2\ell)}}\,\mathcal{S}_{n_{\pi(2\ell+1)}n_{\pi(2\ell)}}\prod_{v=2a+1}^{J}\zeta_{n_{\pi(v)}}\cdot\mathcal{P}_{n_{\pi(v)}}\, e^{i(p-\sum_r n_r q)\cdot X}(z)$$



where $J=\sum_j g_j$, and $\widehat{S}_J = Symm(J)/Z_J$ and $\mathcal{S}_{r,s}[\partial X]\,,\mathcal{P}_n[\partial X]$ of the form

$$\mathcal{S}_{r,s}[\partial X] = \sum_{h=1}^{s}\mathcal{Z}_{h+r}[U_{\partial X}^{(r,\ell)}]\mathcal{Z}_{s-h}[U_{\partial X}^{(s,\ell)}]\,,\;\; \mathcal{P}_n^j = \sum_{h=1}^{n}\frac{\partial^h X^j}{(h-1)!}\mathcal{Z}_{n-h}[U_{\partial X}^{(n,\ell)}]$$

with $\mathcal{Z}_n$ the cycle index polynomial of Symm group S_n of argument $U_{\partial X}^{(n,\ell)}$

$$\mathcal{Z}_n[r_s] = \oint_C \frac{dz}{2\pi i z^{n+1}} e^{\sum_s r_s z^s/s}\,, \qquad U_{\partial X}^{(n,\ell)} = -in\frac{q\cdot\partial^\ell X}{(\ell-1)!}$$

Quite remarkably it was shown that the coherent superposition of creation operators acting on the vacuum state is still an exponential vertex operator!

$$e^{\sum_n \lambda_n\cdot A_{-n}}|0,p\rangle = |\mathcal{C}(\{\lambda\}_n,p,q)\rangle$$

which is coherent under the action of the annihilation operators

$$A^j_m|\mathcal{C}(\{\lambda\}_n,p,q)\rangle = \lambda^j_m|\mathcal{C}(\{\lambda\}_n,p,q)\rangle$$

and has the following compact form

$$\mathcal{V}_{\mathcal{C}}(\{\zeta\}_n;z) = \exp\left\{\sum_{r,s}^{1,\infty}\frac{\zeta_r\cdot\zeta_s}{2}\,\mathcal{S}_{r,s}\,e^{-i(r+s)q\cdot X} + \sum_n^{1,\infty}\zeta_n\cdot\mathcal{P}_n\,e^{-inq\cdot X}\right\}e^{ip\cdot X}(z)$$

This is the generating VO of all the possible VOs of the string spectrum!

$$\prod_n(\lambda_n\cdot A_{-n})^{g_n}|0,p\rangle \Rightarrow \prod_n\left(\zeta_n\frac{\partial}{\partial\zeta'_n}\right)^{g_n}\mathcal{V}_{\mathcal{C}}(\{\zeta'\}_n;z)$$

Chaos in HES interactions : general picture

HES

STRING
(also FRtj)

$|HES\rangle_N \simeq \sum_J \prod_n \mathcal{O}_n^{g_n}$

State

$|STRING\rangle_N \simeq \mathcal{O}_1^N$

$\mathcal{A}_{HES} \simeq \prod_n f_n^{g_n}(\{\sigma\})$

Interaction

$\mathcal{A}_{STRING} \simeq f_1^N(\{\sigma\})$

Erratic function !

$\{\sigma\}$ **Kinematical parameters**

Smooth function

In order to measure chaos one needs a strategy to parametrize HES processes

Chaos in HES interactions : analysis strategy

- The main observable is the amplitude $\mathcal{A}_{HES}(\{\sigma\})$ which is erratic
- in particular the amplitude, in the $\{\sigma\}$ range, presents many extrema
- considering for simplicity only one variable σ, it is useful to map k extrema in z_k zeros, taking the logarithmic derivative

$$F_{\mathcal{A}}(\sigma) = \frac{d \log \mathcal{A}_{HES}(\sigma)}{d\sigma}\,, \quad F_{\mathcal{A}}(\sigma) = 0 \Rightarrow \sigma = z_k$$

- introducing the ratio of consecutive spacings of zeros

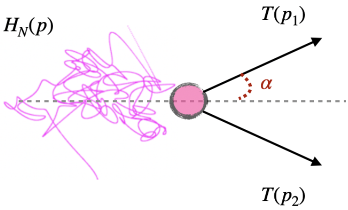
$$r_k = \frac{z_{k+1} - z_k}{z_{k-1} - z_k}$$

- the distribution of r_k will be the discriminant of the chaotic nature, in agreement with RMT distributions such as joint distributions of β-ensambles

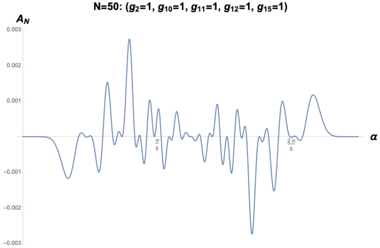
$$f^{(\beta)}_{joint}(r) = \frac{3^{\frac{3}{2}(1+\beta)}\Gamma(1+\beta/2)^2}{2\pi\Gamma(1+\beta)} \frac{(r+r^2)^\beta}{(1+r+r^2)^{1+\frac{3}{2}\beta}}\,, \quad \beta = (1,2,4) \rightarrow G(O,U,S)E$$

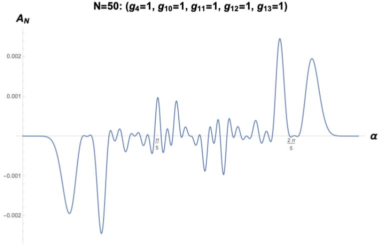
Chaos in HES interactions : $H_N \to T + T$

$$\mathcal{A}(H_N, T, T) \simeq \prod_n \left(\sin(n\pi \cos^2 \alpha)\right)^{g_n}$$



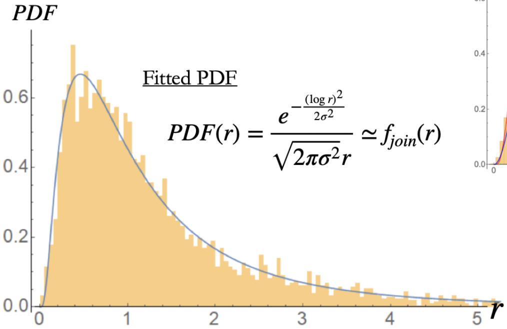
The amplitude is erratic respect to the emission angle α, which is the only kinematic freedom of the amplitude



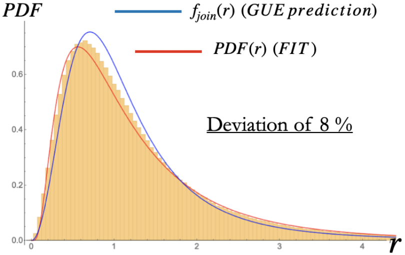


Chaotic behavior of $H_N \to T + T$: results

$$F(\alpha) \equiv \frac{d}{d\alpha} \log \mathcal{A} = J \cot \alpha - \frac{\pi}{2} \sin \alpha \sum_{m=1}^{\infty} m n_m \left[\cot(\pi m \cos^2 \frac{\alpha}{2}) \right]^{n_m}$$



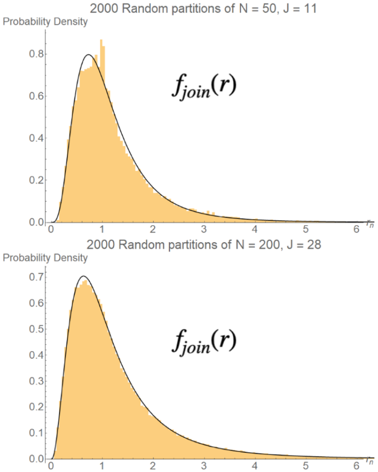
Single state of N=15000 with J=419



From the fitted distribution one can conclude that the HES decay amplitude is chaotic !

$$F(\alpha) \equiv \frac{d}{d\alpha} \log \mathcal{A} = J \cot \alpha - \frac{1}{2} \sin \alpha \sum_{n=1}^{\infty} g_n \sum_{k=1}^{n-1} \frac{n}{n - k - n \cos^2 \frac{\alpha}{2}}$$

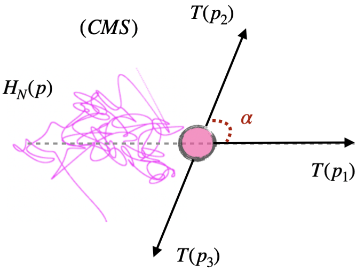
The result is improved if the full form of the amplitude is considered

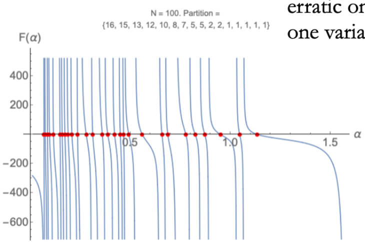


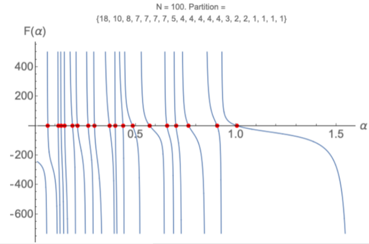
Chaos in HES interactions : $H_N + T \to T + T$

$$\mathcal{A}^{H_N\to 3T}_{Regge} \simeq \prod_n \left\{ \frac{1}{\Gamma(n)} \Gamma\left(n - \frac{n}{\sin\alpha + 1}\right) \Gamma\left(\frac{n}{\sin\alpha+1}\right) \sin\left(\frac{n\pi}{\sin\alpha+1}\right) \right\}^{g_n} \mathcal{A}^{4T}_{Regge}$$

In the Regge regime the amplitude is erratic only in one variable







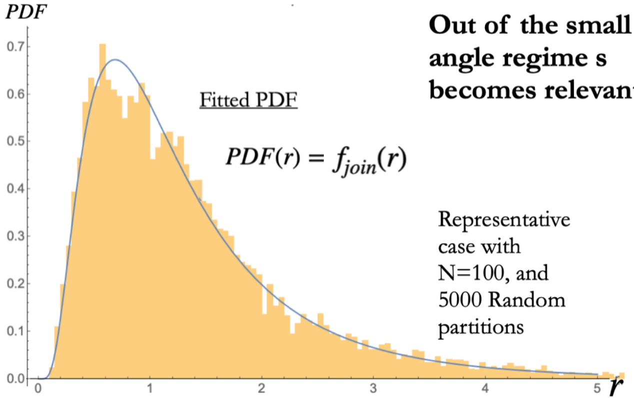
Log derivative analysed:

$$F(\alpha) = -\frac{\cos\alpha}{1+\sin\alpha} \sum_{n=1}^{\infty} g_n \sum_{k=1}^{n-1} \frac{n}{k + (k-n)\sin\alpha}$$

Chaotic behavior of $H_N + T \to T + T$: results

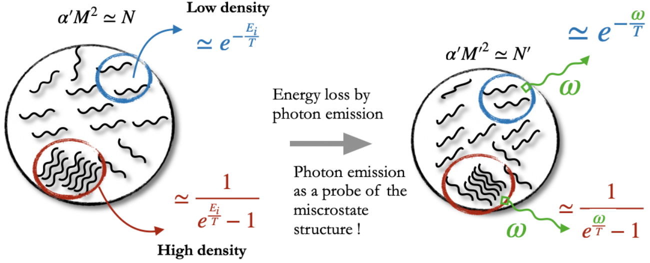
Same distribution ! ***Agreement With GUE !***

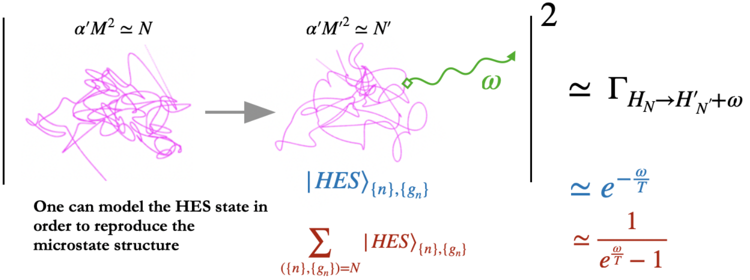
What about the total energy s ?



Out of the small angle regime s becomes relevant ?

Thermalization in HES : general picture





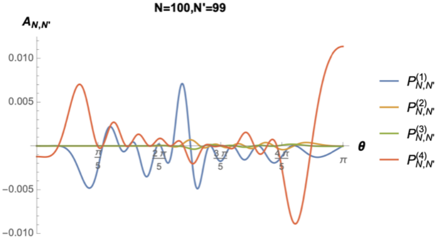
$H_N \to H_{N'} + T$: decay amplitude

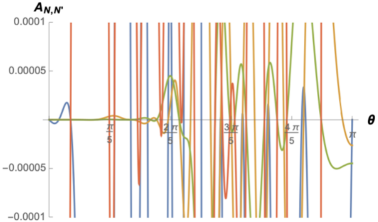
The most general amplitude can be generated by coherent string states !

$$\mathcal{A} = \prod_{n,a} \zeta_n^{(a)} \cdot \frac{\partial}{\partial J_n^{(a)}} \zeta'^{(a)}_n \cdot \frac{\partial}{\partial J'^{(a)}_n} \mathcal{A}_{\text{gen}} \Big|_{J=J'=0}$$

where the generating amplitude can be written in the compact form

$$\mathcal{A}_{\text{gen}} = \exp\left(\sum_{n,a} \left(J_n^{(a)} \cdot \mathcal{V}_n + J'^{(a)}_n \cdot \mathcal{V}'_n \right) + \sum_{n,m,a,b} \left(J_n^{(a)} \cdot J_m^{(b)} \mathcal{W}_{n,m} + J'^{(a)}_n \cdot J'^{(b)}_m \mathcal{W}'_{n,m} + J_n^{(a)} \cdot J'^{(b)}_m \mathcal{M}_{n,m} \right) \right)$$





	$N = \sum_n n g_n$	$N' = \sum_{n'} n' g_{n'}$
$P^{(1)}_{N,N'}$	$g_{30} = 1,\ g_{51} = 1,\ g_{19} = 1$	$g_{49} = 1,\ g_{50} = 1$
$P^{(2)}_{N,N'}$	$g_{30} = 1,\ g_{20} = 1,\ g_{10} = 1,\ g_5 = 1,\ g_{45} = 1$	$g_{99} = 1$
$P^{(3)}_{N,N'}$	$g_{14} = 2,\ g_{20} = 1,\ g_{10} = 1,\ g_5 = 1,\ g_{45} = 1$	$g_{30} = 1,\ g_{69} = 1$
$P^{(4)}_{N,N'}$	$g_{49} = 1,\ g_{51} = 1$	$g_{30} = 1,\ g_{69} = 1$

Amplitude very erratic, not only but also chaotic ! Same structure of previous cases

$H_N \to H_{N'} + T$: decay rate

The decay rate is computed by averaging over polarizations the mod square of the amplitude

$$\Gamma \propto \sum_{pol} |\mathcal{A}|^2 \,, \quad \mathcal{A} = \mathcal{A}(\{\zeta\}_{n,g_n}, \{\zeta'\}_{n,g_n})$$

quite remarkably one can use only the completeness relation of λ_n polarizations :

$$\zeta^M = \lambda^\mu V^M_\mu(p,q) \Rightarrow \sum_{pol} \zeta^M \zeta^{*N} = \eta^{\mu\nu} V^M_\mu(p,q) V^{*N}_\nu(p,q)$$

and the projector is given by

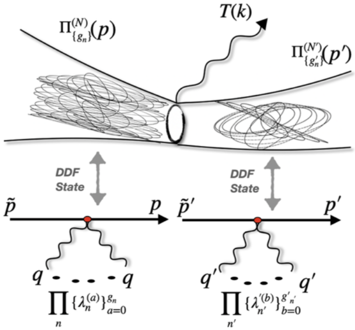
$$\Pi^{M\,N}(p,q) = \eta^{M\,N} - 2p^{(M}q^{N)} + p^2 q^M q^N$$

with properties

$$\Pi^2 = \Pi \,, \quad \Pi^M \cdot p = \Pi^M \cdot q = 0 \,, \quad Tr\,\Pi = D-2$$

Thermal emission : setup

$$\left|\mathcal{A}_{\Pi^{N}_{\{g_n\}},\Pi^{N'}_{\{g_{n'}\}}}\right|^2 \simeq \prod_{n=1}^{N}\left(\frac{\left(1-n(1-E_k/\sqrt{2N})\right)_{n-1}}{\Gamma(n)}\right)^{2g_n}\prod_{n'=1}^{N'}\left(\frac{\left(1-n'(1+E_k/\sqrt{2N})\right)_{n'-1}}{\Gamma(n')}\right)^{2g'_{n'}}$$



-) Compact expression of the decay rate as a function of all the possible decay channels
-) Freedom to model the HES states due to the DDF construction
-) simple 3-pt kinematics parametrized respect to the DDF reference momenta q, q' and polarizations

$$p=\sqrt{2N-2}(1,\vec{0})\,,\quad p'=-(E',\omega\sin\theta,\omega\cos\theta,\vec{0})\,,\quad k=(-E_k,\omega\sin\theta,\omega\cos\theta,\vec{0})$$

$$q=-\frac{(1,0,1,\vec{0})}{\sqrt{2N-2}}\,,\quad \lambda=\frac{(0,1,0,\vec{\Lambda})}{\sqrt{1+|\vec{\Lambda}|^2}}\,;\quad q'=-\frac{(1,0,1,\vec{0})}{\omega\cos\theta-E'}\,,\quad \lambda'=\frac{(0,1,0,\vec{\Lambda}')}{\sqrt{1+|\vec{\Lambda}'|^2}}$$

In the limit in which the energy loss of the decaying string is soft

$$\left|\mathcal{A}_{\Pi^N_{\{g_n\}},\Pi^{N'}_{\{g_{n'}\}}}\right|^2 = \widetilde{\mathcal{N}}^2_{\{g_n\}}\widetilde{\mathcal{N}'}^2_{\{g'_{n'}\}}\, e^{-\mathcal{C}_N(\{g_n\},\{g'_{n'}\})\frac{E_k}{T_H}-2\mu_N\left(\{g_n\},\{g'_{n'}\};E_k/T_H\right)}$$

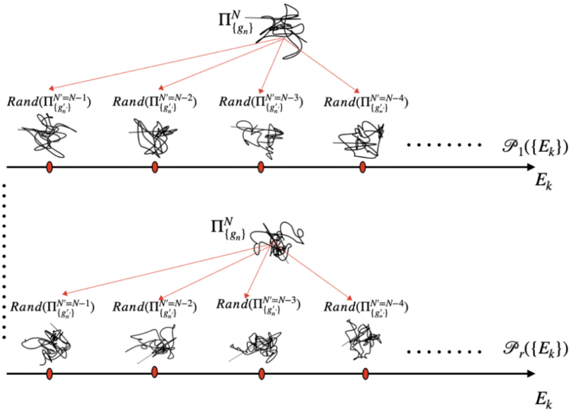
where there are two coefficients that depend non trivially on the microstates of initial and final string

$$\mathcal{C}_N(\{g_n\},\{g'_{n'}\}) = \frac{2}{\sqrt{N}}\left(\sum_{n=1}^{N} g_n n\log n - \sum_{n'=1}^{N'} g'_{n'} n'\log n'\right)$$

and

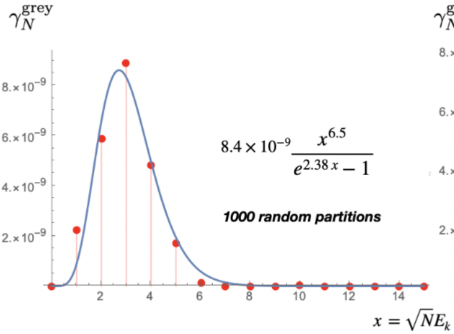
$$\mu_N\left(\{g_n\},\{g'_{n'}\};\frac{E_k}{T_H}\right) = \sum_{n=1}^{N} g_n \log\Gamma\left(1-\frac{n}{\sqrt{N}}\frac{E_k}{T_H}\right)+\sum_{n'=1}^{N'} g'_{n'}\log\Gamma\left(1+\frac{n'}{\sqrt{N}}\frac{E_k}{T_H}\right).$$

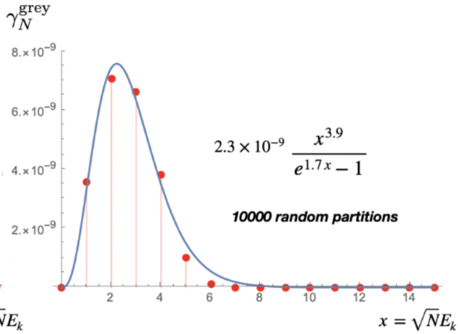
Random spectrum : graybody emission



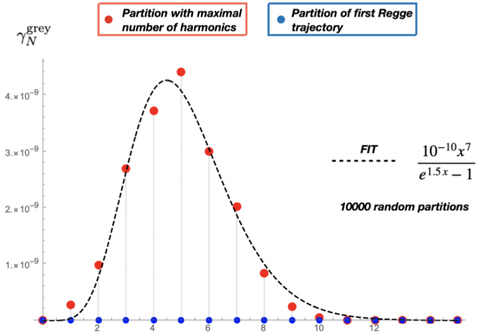
$$\gamma_N^{\text{grey}}(E_k/T_{eff}) = C_N(\{g_n\}.\{g'_{n'}\})\frac{\left(\frac{E_k}{T_{eff}}\right)^{r_N(\{g_n\},\{g'_{n'}\})}}{e^{\nu_N(\{g_n\},\{g'_{n'}\})\frac{E_k}{T_{eff}}}-1}$$

examples of greybody spectra

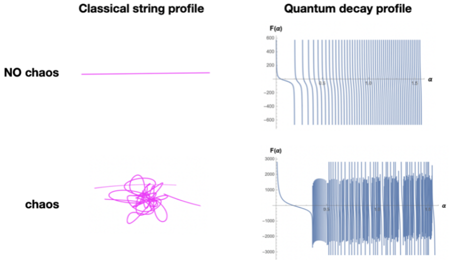




$$T_{eff} = T_H / \sqrt{N}$$



Comparison between the spectra obtained for states of the leading Regge trajectory and states with the maximal number of harmonics (subleading trajectories)



Comparison between single harmonic states (including the leading Regge trajectory) and subleading trajectories.

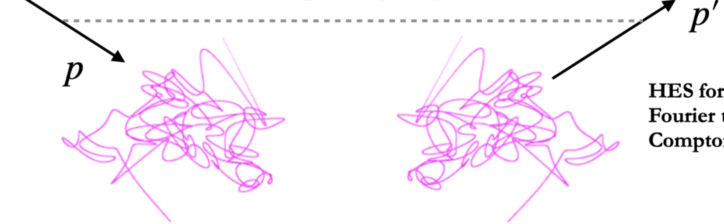
Probing the horizon of HES : the HES form factor

From the Compton amplitude one can extract the form factor of the heavy target (like DIS) and study its size as well as its shape.

size $\propto \sqrt{N}$ **Random walk picture of HES**

shape $\propto F(\{n\}, \{g_n\})$ **The shape can be modeled by taking linear combinations of states at fixed level N**

$$t = -q^2 = (p - p')^2$$



HES form factor from the Fourier transform of the Compton amplitude

HES form factor

In the simplest case, where $P_N = 1_N \Rightarrow (\lambda \cdot A_{-1})^N$

- the Compton amplitude is given by

$$\mathcal{A}_{\text{Comp}}^{T+1_N \Rightarrow 1_N+T} \propto B(\alpha(s), \alpha(t))\, L_N(q^2/2)$$

- in the large N limit at small transfer momentum

$$\mathcal{A}_{\text{Comp}}^{T+1_N \Rightarrow 1_N+T} \propto B(\alpha(s), \alpha(t))\, J_0(q\sqrt{N})$$

- finally the relevant factor $J_0(q\sqrt{N})$ can be Fourier transformed in space

$$J_0(q\sqrt{N}) = \frac{1}{2\pi\sqrt{N}} \int d\vec{X}\, e^{iq\cdot X} \delta(\vec{X}_{\parallel} - \sqrt{N})\delta(\vec{X}_{\perp})$$

Size $\propto \sqrt{N} = \log d_N \propto$ entropy, what about the shape ?

- Compute the Compton amplitude with $1_N \Rightarrow \sum_{P_N \in N} P_N$ (Work in progress)

Conclusion

Chaos in HES

- We have seen how the erratic nature of HES interactions generates the chaotic behavior.

Thermalization in HES decay

- From the HES decay we have seen how to extract a Maxwell-Boltzmann behavior for a single microstate interaction
- the temperature turns out to be $T = (\sqrt{N\,\alpha'})^{-1}$ (Hawking like behavior)

HES horizon

- From the simplest Compton amplitude we have seen that the from factor of the HES $\propto \sqrt{N}$ in agreement with the HES entropy
- The shape is sensitive to the microstate structure and maybe there are chaotic effects, as a natural extension of the chaotic behavior observed.

Thanks for your attention !